

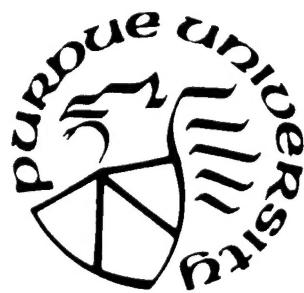
SELECTING THE MOST RELIABLE POISSON  
POPULATION PROVIDED IT IS BETTER THAN A CONTROL:  
A NONPARAMETRIC EMPIRICAL BAYES APPROACH

by

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**DEPARTMENT OF STATISTICS**

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**Abstract**

We study the problem of selecting the most reliable Poisson population from among  $k$  competitors provided it is better than a control using the nonparametric empirical Bayes approach. An empirical Bayes selection procedure is constructed based on the isotonic regression estimators of the posterior means of failure rates associated with the  $k$  Poisson populations. The asymptotic optimality of the empirical Bayes selection procedure is investigated. Under certain regularity conditions, we have shown that the proposed empirical Bayes selection procedure is asymptotically optimal and the associated Bayes risk converges to the minimum Bayes risk at a rate of order  $O(\exp(-cn))$  for some  $c > 0$ , where  $n$  denotes the number of historical data at hand when the present selection problem is considered.

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## 1. Introduction:

In the research and development stage, an experimenter often confronts the problem of selecting the most reliable design from among several competing designs. Usually, the populations that are compared are the life length distributions of the designs. The most reliable design is defined as the one with the longest mean life. The problem of selecting the most reliable design has been studied in the literature. The readers are referred to Gupta and Panchapakesan (1988) for a comprehensive survey of selection procedures in reliable models.

Consider  $k$  types of competing designs  $\pi_1, \dots, \pi_k$ , which are put on life tests. Suppose in case of a failure, the failed design is immediately replaced by the same type of design. It is assumed that the failure times of type  $i$  design are exponentially distributed with an unknown failure rate  $\theta_i$ , and these failure times are mutually independently distributed. Let  $\theta_{[1]} \leq \dots \leq \theta_{[k]}$  denote the ordered values of the failure rate  $\theta_1, \dots, \theta_k$ . The design associated with the smallest failure rate  $\theta_{[1]}$  is called the most reliable design. Over a time period, let  $X_i$  denote the number of failures of type  $i$  design. Then,  $X_i$  follows a Poisson distribution with occurrence rate  $\theta_i$ . With this sampling scheme, Dixon and Bland (1971) derived a Bayes solution to the problem of ranking the failure rates  $\theta_1, \dots, \theta_k$ . Gupta, Leong and Wong (1979) have developed a subset selection procedure for selecting a subset containing the most reliable design. Alam (1971) and Alam and Thompson (1973) have also studied selection procedures for selecting the most reliable design based on inverse sampling observations.

Let  $\theta_0$  be a specified standard or control level. Design  $\pi_i$  is said to be better than the control  $\theta_0$  and acceptable if  $\theta_i < \theta_0$ ; otherwise,  $\pi_i$  is said to be bad and should be excluded. In many practical situations, an experimenter makes a selection only when the most reliable design is better than the control  $\theta_0$ . For example, let  $\theta_0$  be the failure rate of the currently used design. An experimenter may make a selection from among the  $k$  competitors and replace the currently used design by the newly selected one provided the newly selected design is better than the level  $\theta_0$ . Otherwise, the experimenter may select none and continue using the current design.

Consider a situation in which one will be dealing with repeated independent selection

problems. In such instances, it is reasonable to formulate the component problem in the sequence as a Bayes decision problem with respect to an unknown prior distribution on the parameter space. One then uses the accumulated historical data to improve the decision procedures at each stage. This is the so-called empirical Bayes approach (see Robbins (1956, 1964)). Empirical Bayes procedures have been derived for subset selection goals by Deely (1965). For recent developments of empirical Bayes selection procedures on the research area of ranking and selection, the readers are referred to Gupta and Liang (1991), Gupta, Liang and Rau (1994, 1995), and the references quoted there.

In this paper, we study the problem of selecting the most reliable Poisson population provided it is better than a control using the nonparametric empirical Bayes approach. The paper is organized as follows. The selection problem is formulated in Section 2. For a linear loss, we derive a Bayes selection procedure, which is based on the posterior means of the failure rates  $\theta_i, i = 1, \dots, k$ . The empirical Bayes framework of the selection problem is described in Section 3. Based on the isotonic regression estimators of the posterior means of the failure rates, an empirical Bayes selection procedure is constructed. We study the asymptotic optimality of this procedure in Section 4. It is shown that under certain regularity conditions, the proposed empirical Bayes selection procedure is asymptotically optimal and its Bayes risk converges to the minimum Bayes risk at a rate of order  $O(\exp(-cn))$  for some positive number  $c$ , where  $n$  denotes the number of the accumulated historical data at hand when the present selection problem is considered.

## 2. Formulation of the Selection Problem and a Bayes Selection Procedure

Consider  $k (\geq 2)$  independent Poisson populations  $\pi_1, \dots, \pi_k$ , with unknown occurrence rates  $\theta_1, \dots, \theta_k$ , respectively. Let  $\theta_{[1]} \leq \dots \leq \theta_{[k]}$  denote the ordered values of the parameters  $\theta_1, \dots, \theta_k$ . It is assumed that the exact pairing between the ordered and the unordered parameters is unknown. The occurrence rate  $\theta_i$  may be viewed as the failure rate of type  $i$  design, as described previously. Therefore, in the following, a population  $\pi_i$  with  $\theta_i = \theta_{[1]}$  is considered as the most reliable population. Let  $\theta_0$  be a known pre-specified standard level. A population  $\pi_i$  with  $\theta_i < \theta_0$  is said to be better than the control  $\theta_0$  and is acceptable. Otherwise, population  $\pi_i$  is said to be bad and not acceptable. Our goal is to derive empirical Bayes procedures to select the most reliable Poisson population which

should also be better than the control  $\theta_0$ . If there is no such population, we select none and exclude all  $k$  competitors as bad.

Let  $\Omega = \{\underline{\theta} = (\theta_1, \dots, \theta_k) \mid \theta_i > 0, i = 1, \dots, k\}$  be the parameter space and let  $\mathcal{A} = \{a = (a_0, a_1, \dots, a_k) \mid a_i = 0, 1; i = 0, 1, \dots, k, \text{ and } \sum_{i=0}^k a_i = 1\}$  be the action space. For an action  $a$ , when  $a_i = 1$  for some  $i = 1, \dots, k$ , it means that population  $\pi_i$  is selected as the most reliable population and considered to be better than the control  $\theta_0$ ; when  $a_0 = 1$ , it means that all the  $k$  populations are excluded as bad compared with the control  $\theta_0$  and no selection is made. We consider the following linear loss : For  $\underline{\theta} \in \Omega$  and  $a \in \mathcal{A}$ .

$$L(\underline{\theta}, a) = \sum_{i=0}^k a_i \theta_i - \min(\theta_{[1]}, \theta_0). \quad (2.1)$$

For each  $i = 1, \dots, k$ , let  $X_i$  denote a random observation arising from a Poisson population  $\pi_i$  with occurrence rate  $\theta_i$ . Thus, conditioning on  $\theta_i$ ,  $X_i$  has a probability function  $f_i(x|\theta_i)$ , where

$$f_i(x|\theta_i) = \exp(-\theta_i) \theta_i^x / x!, \quad x = 0, 1, 2, \dots \quad (2.2)$$

It is assumed that  $\theta_i$  is a realization of a random variable  $\Theta_i$  which has an unknown prior distribution  $G_i$  over  $(0, \infty)$ . The random vectors  $(X_i, \Theta_i), i = 1, \dots, k$ , are assumed to be mutually independent.

Let  $\mathcal{X}$  be the sample space generated by  $\underline{X} = (X_1, \dots, X_k)$ . A selection procedure  $\underline{\delta} = (\delta_0, \delta_1, \dots, \delta_k)$  is defined to be a mapping from the sample space  $\mathcal{X}$  into the product space  $[0, 1]^{k+1}$ , such that for each  $\underline{x} \in \mathcal{X}$ , the function  $\underline{\delta}(\underline{x}) = (\delta_0(\underline{x}), \delta_1(\underline{x}), \dots, \delta_k(\underline{x}))$  satisfies that  $0 \leq \delta_i(\underline{x}) \leq 1, i = 0, 1, \dots, k$  and  $\sum_{i=0}^k \delta_i(\underline{x}) = 1$ . That is, for each  $i = 1, \dots, k$ ,  $\delta_i(\underline{x})$  is the probability of selecting population  $\pi_i$  as the most reliable population and considering  $\pi_i$  to be better than the control  $\theta_0$ ; and  $\delta_0(\underline{x})$  is the probability of excluding all the  $k$  populations as bad and hence selecting none.

Let  $\mathcal{C}$  be the class of all selection procedures. For each  $\underline{\delta} \in \mathcal{C}$ , let  $R(G, \underline{\delta})$  denote its associated Bayes risk, where  $G(\underline{\theta}) = \prod_{i=1}^k G_i(\theta_i)$ . Then  $R(G) = \inf_{\underline{\delta} \in \mathcal{C}} R(G, \underline{\delta})$  is the minimum Bayes Risk among the class  $\mathcal{C}$ . A selection procedure, say  $\underline{\delta}_G$ , such that  $R(G, \underline{\delta}_G) = R(G)$  is called a Bayes selection procedure. We consider only these priors for which  $\int_0^\infty \theta dG_i(\theta) <$

$\infty$  for each  $i = 1, \dots, k$  so that for each selection procedure  $\underline{\delta}$ ,  $R(G, \underline{\delta})$  is always finite, which insures the selection problem to be meaningful.

Based on the preceding statistical model, the Bayes risk associated with the selection procedure  $\underline{\delta}$  is:

$$\begin{aligned} R(G, \underline{\delta}) &= \sum_{\underline{x} \in \mathcal{X}} \sum_{i=0}^k \delta_i(\underline{x}) \int_{\Omega} \theta_i f(\underline{x} | \theta) dG(\theta) - C \\ &= \sum_{\underline{x} \in \mathcal{X}} \left[ \sum_{i=0}^k \delta_i(\underline{x}) \psi_i(x_i) + \delta_0(\underline{x}) \theta_0 \right] f(\underline{x}) - C \end{aligned} \quad (2.3)$$

where

$$f(\underline{x} | \theta) = \prod_{i=1}^k f_i(x_i | \theta_i),$$

$$f(\underline{x}) = \prod_{i=1}^k f_i(x_i),$$

$$f_i(x_i) = \int f_i(x_i | \theta_i) dG_i(\theta) = h_i(x_i) (x_i!)^{-1}$$

:the marginal probability function of  $X_i$ ,

$$h_i(x_i) = \int \exp(-\theta) \theta^{x_i} dG_i(\theta),$$

$$C = \int_{\Omega} \min(\theta_{[1]}, \theta_0) dG(\theta) \text{ and}$$

$$\psi_i(x_i) = E[\Theta_i | X_i = x_i] = h_i(x_i + 1) / h_i(x_i). \quad (2.4)$$

Note that  $\psi_i(x_i)$  is the posterior mean of  $\Theta_i$  given  $X_i = x_i$ .

From (2.3), a Bayes selection procedure  $\underline{\delta}_G = (\delta_{G0}, \dots, \delta_{Gk})$  can be obtained as follows:  
For each  $\underline{x} \in \mathcal{X}$ , let

$$A(\underline{x}) = \{i = 0, 1, \dots, k | \psi_i(x_i) = \min_{0 \leq j \leq k} \psi_j(x_j)\}, \quad (2.5)$$

where  $\psi_0(x_0) \equiv \theta_0$ . Define

$$i_G \equiv i_G(\underline{x}) = \min\{i | i \in A(\underline{x})\}, \quad (2.6)$$

and for each  $i = 0, \dots, k$ , define

$$\delta_{Gi}(\underline{x}) = \begin{cases} 1, & \text{if } i = i_G, \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

From (2.3), one can see that  $\delta_G$  is a Bayes selection procedure. Also, it should be noted that any selection procedure  $\delta$  satisfying  $\sum_{i \in A(\underline{x})} \delta_i(\underline{x}) = 1$  for each  $\underline{x} \in \mathcal{X}$  is always a Bayes selection procedure.

The minimum Bayes risk is  $R(G, \delta_G)$  where

$$\begin{aligned} R(G, \delta_G) &= \sum_{\underline{x} \in \mathcal{X}} \left[ \sum_{i=0}^k \delta_{Gi}(\underline{x}) \psi_i(x_i) \right] f(\underline{x}) - C \\ &= \sum_{\underline{x} \in \mathcal{X}} \psi_{i_G}(x_{i_G}) f(\underline{x}) - C \end{aligned} \quad (2.8)$$

### 3. Empirical Bayes Selection Procedures

It should be noted that the Bayes selection procedure  $\delta_G$  depends on the prior distribution  $G$ . Since  $G$  is unknown, it is not possible to implement the Bayes procedure  $\delta_G$  for the selection problem at hand. In the following, it is assumed that certain historical data from each of the  $k$  populations are available. In such a situation, the empirical Bayes approach is adopted.

#### 3.1. Empirical Bayes Framework

For each  $i = 1, \dots, k$ , let  $(X_{ij}, \Theta_{ij}), j = 1, 2, \dots$  be random vectors associated with population  $\pi_i$ , where  $X_{ij}$  is observable while  $\Theta_{ij}$  is unobservable. Note that  $\Theta_{ij}$  stands for the failure rate of the design belonging to  $\pi_i$  at stage  $j$ . It is assumed that  $\Theta_{ij}$  has a prior distribution  $G_i$ , for all  $j = 1, 2, \dots$ , and conditioning on  $\Theta_{ij} = \theta_{ij}$ ,  $X_{ij}$  follows a Poisson distribution with occurrence rate  $\theta_{ij}$ ; and  $(X_{ij}, \Theta_{ij}), i = 1, \dots, k, j = 1, 2, \dots$  are mutually independent. At the present stage, say stage  $n + 1$ , let  $\underline{X}_i(n) = (X_{i1}, \dots, X_{in})$  denote the accumulated historical data associated with  $\pi_i$ , and let  $X_i = X_{i,n+1}$  be the present random observation arising from  $\pi_i$ , and  $\theta_i = \theta_{i,n+1}$  be a realization of  $\Theta_{i,n+1}, i = 1, \dots, k$ . Let  $\underline{X}(n) = (\underline{X}_1(n), \dots, \underline{X}_k(n))$  and  $\underline{X} = (X_1, \dots, X_k)$ . At stage  $n + 1$ , we want to select the population associated with  $\theta_{[1]}$  provided that  $\theta_{[1]} < \theta_0$  using the linear loss (2.1).

By (2.5)-(2.7), a natural empirical Bayes selection procedure can be derived as follows. For each  $i = 1, \dots, k$ , based on the accumulated past data  $\underline{X}_i(n)$  and the present observation  $X_i = x_i$ , let  $\psi_{in}(x_i) \equiv \psi_{in}(x_i, \underline{X}_i(n))$  be an empirical Bayes estimator of  $\psi_i(x_i)$ . Then, let

$$A_n(\underline{x}) = \{i = 0, 1, \dots, k | \psi_{in}(x_i) = \min_{0 \leq j \leq k} \psi_{jn}(x_j)\} \quad (3.1)$$

where  $\psi_{0n}(x_0) \equiv \theta_0$ . Define

$$i_n \equiv i_n(\underline{x}) = \min\{i | i \in A_n(\underline{x})\}. \quad (3.2)$$

Analogous to (2.7), an empirical Bayes selection procedure  $\underline{\delta}_n = (\delta_{n0}, \dots, \delta_{nk})$  can be obtained as follows: For each  $\underline{x} \in \mathcal{X}$ ,

$$\delta_{ni}(\underline{x}) = \begin{cases} 1 & \text{if } i = i_n, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

Let  $R(G, \underline{\delta}_n | \underline{X}(n))$  denote the conditional Bayes risk of the selection procedure  $\underline{\delta}_n$  given  $\underline{X}(n)$  and let  $R(G, \underline{\delta}_n)$  be the overall Bayes risk of  $\underline{\delta}_n$ . That is,

$$R(G, \underline{\delta}_n | \underline{X}(n)) = \sum_{\underline{x} \in \mathcal{X}} \psi_{i_n}(x_{i_n}) f(\underline{x}) - C, \quad (3.4)$$

and

$$R(G, \underline{\delta}_n) = E_{\underline{X}(n)} R(G, \underline{\delta}_n | \underline{X}(n)), \quad (3.5)$$

where the expectation  $E_{\underline{X}(n)}$  is taken with respect to the probability measure generated by  $\underline{X}(n)$ .

Note that  $R(G, \underline{\delta}_n | \underline{X}(n)) - R(G, \underline{\delta}_G) \geq 0$  for all  $\underline{X}(n)$  and for all  $n$ , since  $\underline{\delta}_G$  is a Bayes selection procedure. Hence  $R(G, \underline{\delta}_n) - R(G, \underline{\delta}_G) \geq 0$  for all  $n$ . The nonnegative regret Bayes risk  $R(G, \underline{\delta}_n) - R(G, \underline{\delta}_G)$  can be used as a measure of performance of the empirical Bayes selection procedure  $\underline{\delta}_n$ .

A sequence of empirical Bayes selection procedures  $\{\underline{\delta}_n\}_{n=1}^{\infty}$  is said to be asymptotically optimal relative to the prior distribution  $G$  if  $R(G, \underline{\delta}_n) - R(G, \underline{\delta}_G) \rightarrow 0$  as  $n \rightarrow \infty$ . Further,  $\{\underline{\delta}_n\}_{n=1}^{\infty}$  is said to be asymptotically optimal of order  $\{\alpha_n\}_{n=1}^{\infty}$  relative to the

prior distribution  $G$  if  $R(G, \delta_n) - R(G, \delta_G) = O(\alpha_n)$  where  $\{\alpha_n\}_{n=1}^\infty$  is a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ .

In the following, we seek a sequence of empirical Bayes selection procedures possessing the asymptotic optimality.

### 3.2 Construction of Empirical Bayes Selection Procedure

To construct an empirical Bayes selection procedure as described in (3.1)-(3.3), we first need to construct an empirical Bayes estimator for  $\psi_i(x_i)$ . Since  $\psi_i(x_i)$  is increasing in  $x_i$  we desire a monotone estimator for  $\psi_i(x_i)$ .

Based on  $\tilde{X}_i(n)$ , for each  $x = 0, 1, \dots$ , define

$$\begin{cases} f_{in}(x) = \frac{1}{n} \sum_{j=1}^n I_{\{x\}}(X_{ij}), \\ h_{in}(x) = f_{in}(x)/a(x). \end{cases} \quad (3.6)$$

where  $a(x) = 1/(x!)$ . Note that  $f_{in}(x)$  and  $h_{in}(x)$  are unbiased and consistent estimators of  $f_i(x)$  and  $h_i(x)$ , respectively. From (2.4),  $h_{in}(x_i + 1)/h_{in}(x_i)$  is a naive estimator of the posterior mean  $\psi_i(x_i)$ . However, this estimator may not possess the monotonicity property. Thus, we consider an isotonic regression version of  $h_{in}(x_i + 1)/h_{in}(x_i)$ .

Let  $N_{i1} = \min(X_{i1}, \dots, X_{in})$  and  $N_{in} = \max(X_{i1}, \dots, X_{in})$ . For each  $y = 0, 1, \dots$ , define

$$\begin{cases} \Psi_{in}(y) = \sum_{x=0}^y h_{in}(x + 1)a(x + 1), \\ \Psi_i(y) = \sum_{x=0}^y h_i(x + 1)a(x + 1), \\ H_{in}(y) = \sum_{x=0}^y h_{in}(x)a(x + 1), \\ H_i(y) = \sum_{x=0}^y h_i(x)a(x + 1), \end{cases} \quad (3.7)$$

and  $\Psi_{in}(-1) = \Psi_i(-1) = H_{in}(-1) = H_i(-1) = 0$ .

Next, define

$$\Psi_{in}^*(N_{i1}) = \min_{N_{i1} \leq y \leq N_{in}} \left\{ \frac{\Psi_{in}(y) - \Psi_{in}(N_{i1} - 1)}{H_{in}(y) - H_{in}(N_{i1} - 1)} \right\}, \quad (3.8)$$

where  $b/a = \infty$  when  $a = 0$ ; and for each  $x$  between (including)  $N_{i1} + 1$  and  $N_{in}$ , recursively,

define

$$\psi_{in}^*(x) = \min_{x \leq y \leq N_{in}} \left\{ \frac{\Psi_{in}(y) - \sum_{z=N_{i1}}^{x-1} \psi_{in}^*(z) h_{in}(z) a(z+1)}{H_{in}(y) - H_{in}(x-1)} \right\}. \quad (3.9)$$

Note that since  $H_{in}(N_{in}) - H_{in}(x-1) > 0$  for  $N_{i1} \leq x \leq N_{in}$ ,  $\psi_{in}^*(x) < \infty$ .  $\{\psi_{in}^*(x)\}_{x=N_{i1}}^{N_{in}}$  is the isotonic regression of  $\{h_{in}(x+1)/h_{in}(x)\}_{x=N_{i1}}^{N_{in}}$  with random weights  $\{h_{in}(x)a(x+1)\}_{x=N_{i1}}^{N_{in}}$ , see Puri and Singh (1990). Hence,  $\psi_{in}^*(x)$  is nondecreasing in  $x$  for  $x = N_{i1}, \dots, N_{in}$ . By BBBB (1972), for  $N_{i1} < x \leq N_{in}$ ,

$$\begin{aligned} \sum_{z=N_{i1}}^{x-1} \psi_{in}^*(z) h_{in}(z) a(z+1) &\leq \sum_{z=N_{i1}}^{x-1} \frac{h_{in}(z+1)}{h_{in}(z)} h_{in}(z) a(z+1) \\ &= \sum_{z=N_{i1}}^{x-1} h_{in}(z+1) a(z+1) \\ &\leq \Psi_{in}(x-1). \end{aligned} \quad (3.10)$$

Therefore, from (3.8) - (3.10), we have, for each  $x = N_{i1}, \dots, N_{in}$ ,

$$\psi_{in}^*(x) \geq \min_{x \leq y \leq N_{in}} \left\{ \frac{\Psi_{in}(y) - \Psi_{in}(x-1)}{H_{in}(y) - H_{in}(x-1)} \right\}. \quad (3.11)$$

For  $x = 0, 1, \dots, N_{i1} - 1$ , define  $\psi_{in}^*(x) = \psi_{in}^*(N_{i1})$ ; for  $x > N_{in}$ , define  $\psi_{in}^*(x) = \psi_{in}^*(N_{in})$ . Therefore, we see that  $\psi_{in}^*(x)$  is a monotone function of the nonnegative integers  $x$ .

Now, we let  $\delta_n^* = (\delta_{n0}^*, \dots, \delta_{nk}^*)$  be the empirical Bayes selection procedure defined through (3.1)-(3.3) by replacing  $\psi_{in}(x_i)$  by  $\psi_{in}^*(x_i)$ . We also denote its associated  $A_n(x)$  and  $i_n$  by  $A_n^*(x)$  and  $i_n^*$ , respectively. Then, conditioning on  $\underline{X}(n)$ , the conditional Bayes risk of  $\delta_n^*$  is:

$$R(G, \delta_n^* | \underline{X}(n)) = \sum_{\underline{x} \in \mathcal{X}} \psi_{in}^*(x_{i_n^*}) f(\underline{x}) - C, \quad (3.12)$$

and its overall Bayes risk is:

$$R(G, \delta_n^*) = E_{\underline{X}(n)} R(G, \delta_n^* | \underline{X}(n)). \quad (3.13)$$

#### 4. Asymptotic Optimality of $\delta_n^*$

In this section, we evaluate the asymptotic optimality of the empirical Bayes selection procedure  $\delta_n^*$ . We assume the prior distribution  $G_i$  being nondegenerate so that  $\psi_i(x_i)$  is strictly increasing in  $x_i$ . Let

$$\begin{aligned} B_i(1) &= \{x_i | \psi_i(x_i) < \theta_0\}, \\ B_i(2) &= \{x_i | \psi_i(x_i) = \theta_0\} \\ \text{and } B_i(3) &= \{x_i | \psi_i(x_i) > \theta_0\}. \end{aligned}$$

Define

$$\begin{aligned} m_i &= \begin{cases} \sup B_i(1) & \text{if } B_i(1) \neq \emptyset, \\ -1 & \text{otherwise;} \end{cases} \\ M_i &= \begin{cases} \inf B_i(3) & \text{if } B_i(3) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \quad (4.1)$$

By the increasing property of  $\psi_i(x)$ ,  $m_i \leq M_i$ . When  $B_i(3) \neq \emptyset$ ,  $m_i < M_i < \infty$ . We may have either  $M_i = m_i + 1$  for which  $B_i(2) = \emptyset$ , or  $M_i = m_i + 2$  for which  $\psi_i(m_i + 1) = \theta_0$  and  $B_i(2) = \{m_i + 1\}$ .

Let  $E_i$  denote the event that  $N_{i1} = 0$  and  $N_{in} \geq M_i + 2$ ,  $i = 1, \dots, k$ , and  $E = \bigcap_{i=1}^k E_i$ ; let  $E_i^c$  and  $E^c$  denote the complements of the events  $E_i$  and  $E$ , respectively. Also, let  $A_i = \{\underline{x} \in \mathcal{X} | i_G(\underline{x}) = i\}$ ,  $i = 0, 1, \dots, k$ .

##### 4.1 Analysis of Regret Bayes Risk

From (2.8) and (3.12), given  $\underline{X}(n)$ , the conditional regret Bayes risk of  $\delta_n^*$  is:

$$\begin{aligned} &R(G, \delta_n^* | \underline{X}(n)) - R(G, \delta_G) \\ &= \sum_{\underline{x} \in \mathcal{X}} [\psi_{i_n^*}(x_{i_n^*}) - \psi_{i_G}(x_{i_G})] I(E) f(\underline{x}) \\ &+ \sum_{\underline{x} \in \mathcal{X}} [\psi_{i_n^*}(x_{i_n^*}) - \psi_{i_G}(x_{i_G})] I(E^c) f(\underline{x}) \end{aligned} \quad (4.2)$$

where  $I(S)$  denotes the indicator function of the event  $S$ . Now,

$$\begin{aligned}
& \sum_{\tilde{x} \in \mathcal{X}} [\psi_{i_n^*}(x_{i_n^*}) - \psi_{i_G}(x_{i_G})] I(E) f(\tilde{x}) \\
&= \sum_{\tilde{x} \in \mathcal{X}} \sum_{i=0}^k \sum_{j=0}^k I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = j \text{ and } E\} [\psi_i(x_i) - \psi_j(x_j)] f(\tilde{x}) \\
&= \sum_{\tilde{x} \in A_0} \sum_{i=1}^k I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = 0, E\} [\psi_i(x_i) - \theta_0] f(\tilde{x}) \\
&\quad + \sum_{j=1}^k \sum_{\tilde{x} \in A_j} I\{i_n^*(\tilde{x}) = 0, i_G(\tilde{x}) = j, E\} [\theta_0 - \psi_j(x_j)] f(\tilde{x}) \\
&\quad + \sum_{j=1}^k \sum_{i=1}^k \sum_{\tilde{x} \in A_j} I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = j, E\} [\psi_i(x_i) - \psi_j(x_j)] f(\tilde{x}).
\end{aligned} \tag{4.3}$$

On  $A_0$ , for  $i \in A(\tilde{x})$ ,  $\psi_i(x_i) = \theta_0$ . So,

$$\begin{aligned}
& I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = 0, E\} [\psi_i(x_i) - \theta_0] \\
&= I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = 0, i \notin A(\tilde{x}), E\} [\psi_i(x_i) - \theta_0].
\end{aligned} \tag{4.4}$$

For each  $j = 1, \dots, k$ , on  $A_j$ , for  $i \in A(\tilde{x})$ ,  $\psi_i(x_i) = \psi_j(x_j)$ . So,

$$\begin{aligned}
& I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = j, E\} [\psi_i(x_i) - \psi_j(x_j)] \\
&= I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = j, i \notin A(\tilde{x}), E\} [\psi_i(x_i) - \psi_j(x_j)].
\end{aligned} \tag{4.5}$$

Combining (4.2)–(4.6), the regret Bayes risk of  $\delta_n^*$  can be written as:

$$\begin{aligned}
& R(G, \delta_n^*) - R(G, \delta_G) \\
&= \sum_{\tilde{x} \in A_0} \sum_{i=1}^k E_{\tilde{X}(n)} I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = 0, i \notin A(\tilde{x}), E\} [\psi_i(x_i) - \theta_0] f(\tilde{x}) \\
&\quad + \sum_{j=1}^k \sum_{\tilde{x} \in A_j} E_{\tilde{X}(n)} I\{i_n^*(\tilde{x}) = 0, i_G(\tilde{x}) = j, 0 \notin A(\tilde{x}), E\} [\theta_0 - \psi_j(x_j)] f(\tilde{x}) \\
&\quad + \sum_{j=1}^k \sum_{i=1}^k \sum_{\tilde{x} \in A_j} E_{\tilde{X}(n)} I\{i_n^*(\tilde{x}) = i, i_G(\tilde{x}) = j, i \notin A(\tilde{x}), E\} [\psi_i(x_i) - \psi_j(x_j)] f(\tilde{x}) \\
&\quad + E_{\tilde{X}(n)} \left[ \sum_{\tilde{x} \in \mathcal{X}} [\psi_{i_n^*}(x_{i_n^*}) - \psi_{i_G}(x_{i_G})] I(E^c) f(\tilde{x}) \right] \\
&\equiv I_n + II_n + III_n + IV_n.
\end{aligned} \tag{4.6}$$

## 4.2 Propositions

**Proposition 4.1** For  $\underline{x} \in A_0$  and  $i \notin A(\underline{x})$ ,

$$\begin{aligned} & E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = 0, i \notin A(\underline{x}) \text{ and } E\} \\ & \leq d \exp\{-2n[b_i(M_i, \theta_0) \min(1, \theta_0^{-1})/4]^2\}, \end{aligned}$$

where  $b_i(M_i, \theta_0) = [h_i(M_i + 1) - \theta_0 h_i(M_i)]a(M_i + 1)$  and  $d$  is a constant independent of the distribution of  $\underline{X}(n)$ .

**Proof:** For  $\underline{x} \in A_0$  and  $i \notin A(\underline{x})$ , we have  $x_i \geq M_i$  and  $\psi_i(x_i) > \theta_0$ . By definitions of  $i_n^*, i_G$  and the monotonicity of the function  $\psi_{in}^*(x)$ , we obtain the following:

$$\begin{aligned} & E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = 0, i \notin A(\underline{x}) \text{ and } E\} \\ & \leq P\{\psi_{in}^*(x_i) < \theta_0 \text{ and } E\} \\ & \leq P\{\psi_{in}^*(M_i) < \theta_0 \text{ and } E\} \\ & \leq d \exp\{-2n[b_i(M_i, \theta_0) \min(1, \theta_0^{-1})/4]^2\}, \end{aligned}$$

where the last inequality is obtained from Corollary 5.1.  $\square$

**Proposition 4.2** For each  $j = 1, \dots, k$  and for  $\underline{x} \in A_j$ ,

$$E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = 0, i_G(\underline{x}) = j \text{ and } E\} \leq \sum_{z=0}^{m_j} \exp\{-2nc_j^2(z, \theta_0)\},$$

where  $c_j(z, \theta_0) = [h_j(z + 1) - \theta_0 h_j(z)] / (\frac{1}{a(z+1)} + \frac{\theta_0}{a(z)})$ .

**Proof:** For  $\underline{x} \in A_j$ ,  $x_j \leq m_j$  and  $\psi(x_j) < \theta_0$ . Thus, by the monotonicity of  $\psi_{jn}^*(x)$  and by Corollary 5.2, we can obtain

$$\begin{aligned} & E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = 0, i_G(\underline{x}) = j \text{ and } E\} \\ & \leq P\{\psi_{jn}^*(x_j) \geq \theta_0 \text{ and } E\} \\ & \leq P\{\psi_{jn}^*(m_j) \geq \theta_0 \text{ and } E\} \\ & \leq \sum_{z=0}^{m_j} \exp\{-2nc_j^2(z, \theta_0)\}. \end{aligned}$$

For each  $j = 1, \dots, k$ , each  $\underline{x} \in A_j$  and  $i \notin A(\underline{x})$ ,  $\psi_j(x_j) = \min_{0 \leq \ell \leq k} \psi_\ell(x_\ell) < \min(\theta_0, \psi_i(x_i))$ . There are two cases regarding the value of  $x_i$ : either  $x_i \geq M_i$  for which  $\psi_i(x_i) > \theta_0$  or  $x_i \leq M_i - 1$ . Thus,

$$\begin{aligned} & E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}) \text{ and } E\} \\ &= E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \geq M_i \text{ and } E\} \\ & \quad + E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \leq M_i - 1 \text{ and } E\}. \end{aligned}$$

### Proposition 4.3

(a)  $E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \geq M_i \text{ and } E\}$   
 $\leq d \exp\{-2n[b_i(M_i, \theta_0) \min(1, \theta_0^{-1})/4]^2\}.$

(b)  $E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \leq M_i - 1 \text{ and } E\}$   
 $\leq d \exp\{-2n[b_i(x_i, s(x_i, x_j)) \min(1, 1/s(x_i, x_j))/4]^2\}.$   
 $+ \sum_{z=0}^{x_j} \exp\{-2n c_j^2(z, s(x_i, x_j))\},$   
where  $s(x_i, x_j) = [\psi_i(x_i) + \psi_j(x_j)]/2$ .

### Proof:

(a)  $E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \geq M_i \text{ and } E\}$   
 $\leq P\{\psi_{in}^*(x_i) < \theta_0, x_i \geq M_i \text{ and } E\}$   
 $\leq P\{\psi_{in}^*(M_i) < \theta_0 \text{ and } E\}$   
 $\leq d \exp\{-2n[b_i(M_i, \theta_0) \min(1, \theta_0^{-1})/4]^2\},$   
by Corollary 5.1.

(b) Note that  $\psi_j(x_j) < s(x_i, x_j) = [\psi_i(x_i) + \psi_j(x_j)]/2 < \psi_i(x_i)$ . By Corollaries 5.1 and 5.2,

$$\begin{aligned} & E_{\underline{X}(n)} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \leq M_i - 1 \text{ and } E\} \\ & \leq P\{\psi_{in}^*(x_i) - \psi_{jn}^*(x_j) \leq 0 \text{ and } E\} \\ & = P\{[\psi_{in}^*(x_i) - \psi_i(x_i)] - [\psi_{jn}^*(x_j) - \psi_j(x_j)] < -\psi_i(x_i) + \psi_j(x_j) \text{ and } E\} \end{aligned}$$

$$\begin{aligned}
&\leq P\{\psi_{in}^*(x_i) - \psi_i(x_i) < [-\psi_i(x_i) + \psi_j(x_j)]/2 \text{ and } E\} \\
&\quad + P\{\psi_{jn}^*(x_j) - \psi_j(x_j) > [\psi_i(x_i) - \psi_j(x_j)]/2 \text{ and } E\} \\
&= P\{\psi_{in}^*(x_i) < s(x_i, x_j) \text{ and } E\} \\
&\quad + P\{\psi_{jn}^*(x_j) > s(x_i, x_j) \text{ and } E\} \\
&\leq d \exp\{-2n[b_i(x_i, s(x_i, x_j)) \min(1, 1/s(x_i, x_j))/4]^2\} \\
&\quad + \sum_{z=0}^{x_j} \exp\{-2nc_j^2(z, s(x_i, x_j))\}.
\end{aligned}$$

□

**Proposition 4.4**

$$\begin{aligned}
&E_{\tilde{X}(n)}\{[\sum_{\tilde{x} \in \mathcal{X}} [\psi_{in}^*(x_{in}^*) - \psi_{in}(x_{in})] f(\tilde{x})] I(E^c)\} \\
&\leq C_1 \sum_{i=1}^k [\exp\{-n \ln[1 - f_i(0)]^{-1}\} + \exp\{-n \ln[F_i(M_i + 1)]^{-1}\}],
\end{aligned}$$

$$\text{where } C_1 = \theta_0 + \sum_{i=1}^k \int \theta dG_i(\theta).$$

**Proof:**

$$\begin{aligned}
0 &\leq \sum_{\tilde{x} \in \mathcal{X}} [\psi_{in}^*(x_{in}^*) - \psi_{in}(x_{in})] f(\tilde{x}) I(E^c) \\
&\leq \sum_{\tilde{x} \in \mathcal{X}} [\theta_0 + \sum_{i=1}^k \psi_i(x_i)] f(\tilde{x}) I(E^c) \\
&= [\theta_0 + \sum_{i=1}^k \int \theta dG_i(\theta)] I(E^c) \\
&= C_1 I(E^c).
\end{aligned}$$

Therefore,

$$E_{\tilde{X}(n)}\{[\sum_{\tilde{x} \in \mathcal{X}} [\psi_{in}^*(x_{in}^*) - \psi_{in}(x_{in})] f(\tilde{x})] I(E^c)\} \leq C_1 P\{E^c\},$$

where

$$\begin{aligned}
P(E^c) &\leq \sum_{i=1}^k P(E_i^c) \\
&= \sum_{i=1}^k [P\{N_{i1} > 0\} + P\{N_{in} \leq M_i + 1\}] \\
&= \sum_{i=1}^k [[1 - f_i(0)]^n + [F_i(M_i + 1)]^n].
\end{aligned}$$

Hence the result follows.  $\square$

### 4.3 Rate of Convergence

The main result of this paper is regarding the rate of convergence of the regret Bayes risk of the empirical Bayes selection procedure  $\tilde{\delta}_n^*$ . This result is stated as a theorem as follows.

**Theorem 4.1** Let  $\tilde{\delta}_n^*$  be the empirical Bayes selection procedure constructed in Section 3. Suppose that

- (a)  $\int_0^\infty \theta dG_i(\theta) < \infty$  for each  $i = 1, \dots, k$ , and
- (b)  $M_i < \infty$  for each  $i = 1, \dots, k$ .

Then,  $R(G, \tilde{\delta}_n^*) - R(G, \tilde{\delta}_G) = O(\exp\{-\gamma n\})$  for some constant  $\gamma > 0$ .

**Proof:** To prove this theorem, it suffices to investigate the asymptotic behaviors of the four terms  $I_n, II_n, III_n$  and  $IV_n$  given in (4.6).

(I) Let  $\gamma_{1i} = 2[b_i(M_i, \theta_0) \min(1, \theta_0^{-1})/4]^2$  and  $\gamma_1 = \min_{1 \leq i \leq k} \gamma_{1i}$ . Note that  $\gamma_{1i} > 0$  for each  $i = 1, \dots, k$  and therefore  $\gamma_1 > 0$ . By Proposition 4.1,

$$\begin{aligned} I_n &\leq \sum_{\tilde{x} \in A_0} \sum_{i=1}^k d \exp\{-n\gamma_{1i}\} \psi_i(x_i) f(x) \\ &\leq d \exp\{-n\gamma_1\} \sum_{\tilde{x} \in \mathcal{X}} \sum_{i=1}^k \psi_i(x_i) f(x) \\ &= d \exp\{-n\gamma_1\} \left[ \sum_{i=1}^k \int_0^\infty \theta dG_i(\theta) \right]. \end{aligned}$$

(II) Let  $\gamma_{2j} = 2 \min_{0 \leq z \leq m_j} c_j^2(z, \theta_0)$  and  $\gamma_2 = \min_{1 \leq j \leq k} \gamma_{2j}$ . Then  $\gamma_{2j} > 0$  for each  $j = 1, \dots, k$ ,

and therefore,  $\gamma_2 > 0$ . By Proposition 4.2,

$$\begin{aligned}
\Pi_n &\leq \sum_{j=1}^k \sum_{\underline{x} \in A_j} \sum_{z=0}^{m_j} \exp\{-n\gamma_{2j}\} \theta_0 f(\underline{x}) \\
&\leq \theta_0 \exp\{-n\gamma_2\} [\max(m_1, \dots, m_k) + 1] \sum_{j=1}^k \sum_{\underline{x} \in A_j} f(\underline{x}) \\
&\leq \theta_0 \exp\{-n\gamma_2\} [\max(m_1, \dots, m_k) + 1]
\end{aligned}$$

since  $\sum_{j=1}^k \sum_{\underline{x} \in A_j} f(\underline{x}) \leq 1$ .

(III) For each  $i \neq j$ , let  $A_j(i) = \{\underline{x} \in A_j | x_i \leq M_i - 1\}$ ,  $A_j^c(i) = \{\underline{x} \in A_j | x_i \geq M_i\}$ . Thus,

$III_n$

$$\begin{aligned}
&= \sum_{j=1}^k \sum_{i=1}^k \sum_{\underline{x} \in A_j(i)} E_{\underline{X}^{(n)}} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \leq M_i - 1, E\} [\psi_i(x_i) - \psi_j(x_j)] f(\underline{x}) \\
&+ \sum_{j=1}^k \sum_{i=1}^k \sum_{\underline{x} \in A_j^c(i)} E_{\underline{X}^{(n)}} I\{i_n^*(\underline{x}) = i, i_G(\underline{x}) = j, i \notin A(\underline{x}), x_i \geq M_i, E\} [\psi_i(x_i) - \psi_j(x_j)] f(\underline{x}) \\
&\equiv III_{n1} + III_{n2}.
\end{aligned}$$

For  $\underline{x} \in A_j(i)$ ,  $x_j \leq m_j$ ,  $x_i \leq M_i - 1$ . Let

$$\gamma_{3ij1}(\underline{x}) = 2[b_i(x_i, s(x_i, x_j)) \min(1, 1/s(x_i, x_j))/4]^2.$$

Note that  $\gamma_{3ij1}(\underline{x}) > 0$ . Also,  $\gamma_{3ij1}(\underline{x})$  depends on  $\underline{x}$  only through  $x_i$  and  $x_j$ , for which  $0 \leq x_i \leq M_i - 1$ , and  $0 \leq x_j \leq m_j$ . Therefore,  $\gamma_{3ij1} = \min_{\underline{x} \in A_j(i)} \gamma_{3ij1}(\underline{x}) > 0$ .

Let  $\gamma_{3ij2}(\underline{x}, z) = 2c_j^2(z, s(x_i, x_j))$ ,  $0 \leq z \leq x_j$ . Note that  $\gamma_{3ij2}(\underline{x}, z) > 0$  for each  $0 \leq z \leq x_j$ . Therefore,

$$\gamma_{3ij2}(\underline{x}) \equiv \min_{0 \leq z \leq x_j} \gamma_{3ij2}(\underline{x}, z) > 0, \gamma_{3ij2} \equiv \min_{\underline{x} \in A_j(i)} \gamma_{3ij2}(\underline{x}) > 0,$$

$$\gamma_{3ij} \equiv \min(\gamma_{3ij1}, \gamma_{3ij2}) > 0, \gamma_{3i} = \min_{\substack{1 \leq j \leq k \\ j \neq i}} \gamma_{3ij} > 0 \text{ and } \gamma_3 = \min_{1 \leq i \leq k} \gamma_{3i} > 0.$$

By Proposition 4.3(b),

$$\begin{aligned}
III_{n1} &\leq \sum_{j=1}^k \sum_{i=1}^k \sum_{\substack{\underline{x} \in A_j(i)}} [d \exp\{-n\gamma_3\} + (x_j + 1) \exp\{-n\gamma_3\}] \psi_i(x_i) f(\underline{x}) \\
&\leq [d + \max(m_1, \dots, m_k) + 1] \exp\{-n\gamma_3\} \sum_{j=1}^k \sum_{i=1}^k \sum_{\substack{\underline{x} \in A_j(i)}} \psi_i(x_i) f(\underline{x}) \\
&\leq [d + \max(m_1, \dots, m_k) + 1] \exp\{-n\gamma_3\} k \left[ \sum_{i=1}^k \int_0^\infty \theta dG_i(\theta) \right].
\end{aligned}$$

By Proposition 4.3 (a) and the definition of  $\gamma_{1i}$ ,

$$\begin{aligned}
III_{n2} &\leq \sum_{j=1}^k \sum_{i=1}^k \sum_{\substack{\underline{x} \in A_j^c(i)}} d \exp\{-n\gamma_{1i}\} \psi_i(x_i) f(\underline{x}) \\
&\leq d \exp\{-n\gamma_1\} \sum_{j=1}^k \sum_{i=1}^k \sum_{\substack{\underline{x} \in A_j^c(i)}} \psi_i(x_i) f(\underline{x}) \\
&\leq d \exp\{-n\gamma_1\} k \left[ \sum_{i=1}^k \int_0^\infty \theta dG_i(\theta) \right].
\end{aligned}$$

(IV) Let  $\gamma_{4i} = \min(\ell n[1 - f_i(0)]^{-1}, \ell n[F_i(M_i + 1)]^{-1})$  and  $\gamma_4 = \min_{1 \leq i \leq k} \gamma_{4i}$ .

Then  $\gamma_{4i} > 0$  for each  $i = 1, \dots, k$  and therefore  $\gamma_4 > 0$ . By Proposition 4.4,

$$IV_n \leq 2kC_1 \exp\{-n\gamma_4\},$$

where  $C_1 = \theta_0 + \sum_{i=1}^k \int_0^\infty \theta dG_i(\theta)$ .

Now,  $\gamma \equiv \min(\gamma_1, \gamma_2, \gamma_3, \gamma_4) > 0$ . From the preceding analysis, we have:  $I_n = O(\exp(-\gamma n))$ ,  $II_n = O(\exp(-\gamma n))$ ,  $III_n = O(\exp(-\gamma n))$  and  $IV_n = O(\exp(-\gamma n))$ . Therefore, from (4.6), we conclude that  $R(G, \delta_n^*) - R(G, \delta_G) = O(\exp\{-\gamma n\})$ .

## 5. Preliminary Results

In this section, we introduce certain preliminary results for presenting a concise proofs for the Propositions of Section 4.

Define  $\Delta_{\Psi_{in}}(y) = \Psi_{in}(y) - \Psi_i(y)$ ,  $\Delta_{H_{in}}(y) = H_{in}(y) - H_i(y)$ . Also, let  $F_i(y)$  denote the marginal distribution function of the random variable  $X_i$ , and let  $F_{in}(y)$  be the empirical distribution based on  $X_i(n)$ .

**Lemma 5.1** For  $c > 0$  and  $0 \leq x < N_{in}$ ,

$$(a) \left[ \bigcup_{y=x}^{N_{in}} \{\Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1) < -c\} \right] \subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| > \frac{c}{2} \right\}.$$

$$(b) \left[ \bigcup_{y=x}^{N_{in}} \{\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x-1) \geq c\} \right] \subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| > \frac{c}{2} \right\}.$$

**Proof:** (a) From (3.7), for  $y \geq x \geq 0$ ,

$$\begin{aligned} \Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1) &= \sum_{z=x}^y [h_{in}(z+1) - h_i(z+1)]a(z+1) \\ &= [F_{in}(y+1) - F_i(y+1)] - [F_{in}(x) - F_i(x)]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\bigcup_{y=x}^{N_{in}} \{\Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1) < -c\} \\ &\subset \bigcup_{y=x}^{N_{in}} \{F_{in}(y+1) - F_i(y+1) < -\frac{c}{2} \text{ or } F_{in}(x) - F_i(x) > \frac{c}{2}\} \\ &\subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| > \frac{c}{2} \right\}. \end{aligned}$$

(b) Note that for  $y \geq x \geq 0$

$$\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x) = \sum_{z=x}^y [h_{in}(z) - h_i(z)]a(z) \frac{a(z+1)}{a(z)},$$

where  $\frac{a(z+1)}{a(z)} = \frac{1}{z+1}$ , which is decreasing in  $z$  for  $z = 0, 1, \dots$ , and bounded above by 1. Then by Lemma 3.1 of Gupta and Liang (1991), and following a proof analogous

to that of part (a) of this lemma, we obtain

$$\begin{aligned}
& \bigcup_{y=x}^{N_{in}} \{\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x-1) \geq c\} \\
&= \left\{ \sum_{z=x}^y [h_{in}(z) - h_i(z)]a(z)/(z+1) \geq c \text{ for some } x \leq y \leq N_{in} \right\} \\
&\subset \left\{ \left| \sum_{z=x}^y [h_{in}(z) - h_i(z)]a(z)/(z+1) \right| \geq c \text{ for some } y \geq x \right\} \\
&\subset \left\{ \left| \sum_{z=x}^{\infty} [h_{in}(z) - h_i(z)]a(z) \right| \geq c \text{ for some } y \geq x \right\} \\
&\subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| \geq \frac{c}{2} \right\}.
\end{aligned}$$

Hence the proof is completed.  $\square$

Define  $b_i(x, c) = [h_i(x+1) - ch_i(x)]a(x+1)$ ,  $x = 0, 1, \dots$ , and  $i = 1, \dots, k$ .

**Lemma 5.2** For  $y \geq x \geq 0$  and  $0 < c < \psi_i(x)$ ,

$$[-\Psi_i(y) + \Psi_i(x-1)] + c[H_i(y) - H_i(x-1)] \leq -b_i(x, c) < 0.$$

**Proof:** By the increasing property of  $\psi_i(x)$ ,  $\psi_i(y) \geq \psi_i(x) > c$  for  $y \geq x$ .

From (3.7), for  $y \geq x$ ,

$$\begin{aligned}
& [-\Psi_i(y) + \Psi_i(x-1)] + c[H_i(y) - H_i(x-1)] \\
&= - \sum_{z=x}^y h_i(z) \left[ \frac{h_i(z+1)}{h_i(z)} - c \right] a(z+1) \\
&\leq -h_i(x) \left[ \frac{h_i(x+1)}{h_i(x)} - c \right] a(x+1) \\
&= -b_i(x, c) \\
&< 0
\end{aligned}$$

since  $\frac{h_i(z+1)}{h_i(z)} = \psi_i(z) \geq \psi_i(x) > c$  for all  $z \geq x$  and  $h_i(z) > 0$  for all  $z = 0, 1, \dots$ . Hence the proof is completed.  $\square$

**Lemma 5.3** For each  $x = 0, 1, \dots, m_i + 2$ , and  $c$  such that  $0 < c < \psi_i(x)$ ,

$$\{\psi_{in}^*(x) \leq c \text{ and } E\} \subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| \geq b_i(x, c) \min(1, c^{-1})/4 \right\}.$$

**Proof:** On  $E$ ,  $m_i + 2 < N_{in}$  and  $N_{i1} = 0$ . Thus, from (3.11), for each  $x = 0, 1, \dots, m_i + 2$  and  $0 < c < \psi_i(x)$ , by Lemma 5.2,

$$\begin{aligned}
& \{\psi_{in}^*(x) \leq c \text{ and } E\} \\
& \subset \left\{ \min_{x \leq y \leq N_{in}} \frac{\Psi_{in}(y) - \Psi_{in}(x-1)}{H_{in}(y) - H_{in}(x-1)} \leq c \right\} \\
& \subset \bigcup_{y=x}^{N_{in}} \{[\Psi_{in}(y) - \Psi_{in}(x-1)] - c[H_{in}(y) - H_{in}(x-1)] \leq 0\} \\
& = \left[ \bigcup_{y=x}^{N_{in}} \left\{ \begin{array}{l} [\Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1)] - c[\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x-1)] \\ < [-\Psi_i(y) + \Psi_i(x-1)] + c[H_i(y) - H_i(x-1)] \end{array} \right\} \right] \\
& \subset \left[ \bigcup_{y=x}^{N_{in}} \{[\Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1)] - c[\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x-1)] < -b_i(x, c)\} \right] \\
& \subset \left[ \bigcup_{y=x}^{N_{in}} \left( \{[\Delta_{\Psi_{in}}(y) - \Delta_{\Psi_{in}}(x-1)] \leq -\frac{b_i(x, c)}{2}\} \bigcup \{\Delta_{H_{in}}(y) - \Delta_{H_{in}}(x-1) \geq \frac{b_i(x, c)}{2c}\} \right) \right] \\
& \subset \left\{ \sup_{y \geq 0} |F_{in}(y) - F_i(y)| \geq b_i(x, c) \min(1, c^{-1})/4 \right\},
\end{aligned}$$

where the last inclusion relation is obtained due to Lemma 5.1.

This completes the proof.  $\square$

By Lemma 5.3 and the probability inequality for Kolmogorov-Smirnov distance established by Dvoretzky, Kiefer and Wolfowitz (1956), we establish an exponential-type upper bound for the term  $P\{\psi_{in}^*(x) \leq c \text{ and } E\}$ .

**Corollary 5.1** Under the statement of Lemma 5.3,

$$P\{\psi_{in}^*(x) \leq c \text{ and } E\} \leq d \exp\{-2n[b_i(x, c) \min(1, c^{-1})/4]^2\},$$

where  $d$  is a constant independent of the distribution function  $F_i$ .

**Lemma 5.4** For each  $x = 0, 1, \dots, m_i + 2$ , and a positive value  $c > \psi_i(x)$ ,

$$\{\psi_{in}^* \geq c \text{ and } E\} \subset \{h_{in}(z+1) - ch_{in}(z) \geq 0 \text{ for some } 0 \leq z \leq x\}.$$

**Proof:** On  $E$ ,  $N_{i1} = 0$  and  $N_{in} \geq m_i + 2$  and  $\{\psi_{in}^*(x)\}_{x=0}^{N_{in}}$  is the isotonic regression version of  $\{h_{in}(x+1)/h_{in}(x)\}_{x=0}^{N_{in}}$ . Hence,  $\psi_{in}^*(x) \geq c$  implies that  $h_{in}(z+1)/h_{in}(z) \geq c$  for some  $0 \leq z \leq x$ . Therefore, the result follows immediately.  $\square$

**Lemma 5.5** For a positive value  $c$  such that  $h_i(z+1) - ch_i(z) < 0$ ,

$$P\{h_{in}(z+1) - ch_{in}(z) \geq 0\} \leq \exp\{-2nc_i^2(z, c)\},$$

where  $c_i(z, c) = [h_i(z+1) - ch_i(z)]/[\frac{1}{a(z+1)} + \frac{c}{a(z)}]$ .

**Proof:** Note that  $h_{in}(z+1) - ch_{in}(z) = \frac{1}{n} \sum_{j=1}^n W_{ij}(z)$ , where  $W_{ij}(z) = \frac{I_{\{z+1\}}(X_{ij})}{a(z+1)} - \frac{cI_{\{z\}}(X_{ij})}{a(z)}$ ,  $j = 1, \dots, n$ , are *iid*, bounded random variables such that  $-\frac{c}{a(z)} \leq W_{ij}(z) \leq \frac{1}{a(z+1)}$  and  $E_{\tilde{X}(n)}[W_{ij}(z)] = h_i(z+1) - ch_i(z)$ . Therefore, by Hoeffding's inequality, we have

$$\begin{aligned} & P\{h_{in}(z+1) - ch_{in}(z) \geq 0\} \\ &= P\left\{\frac{1}{n} \sum_{j=1}^n W_{ij}(z) - [h_i(z+1) - ch_i(z)] \geq -[h_i(z+1) - ch_i(z)]\right\} \\ &\leq \exp\{-2nc_i^2(z, c)\}. \end{aligned}$$

□

The following corollary is a direct consequence of Lemmas 5.4 and 5.5.

**Corollary 5.2** Under the statement of Lemma 5.4,

$$P\{\psi_{in}^*(x) \geq c \text{ and } E\} \leq \sum_{z=0}^x \exp\{-2nc_i^2(z, c)\}.$$

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